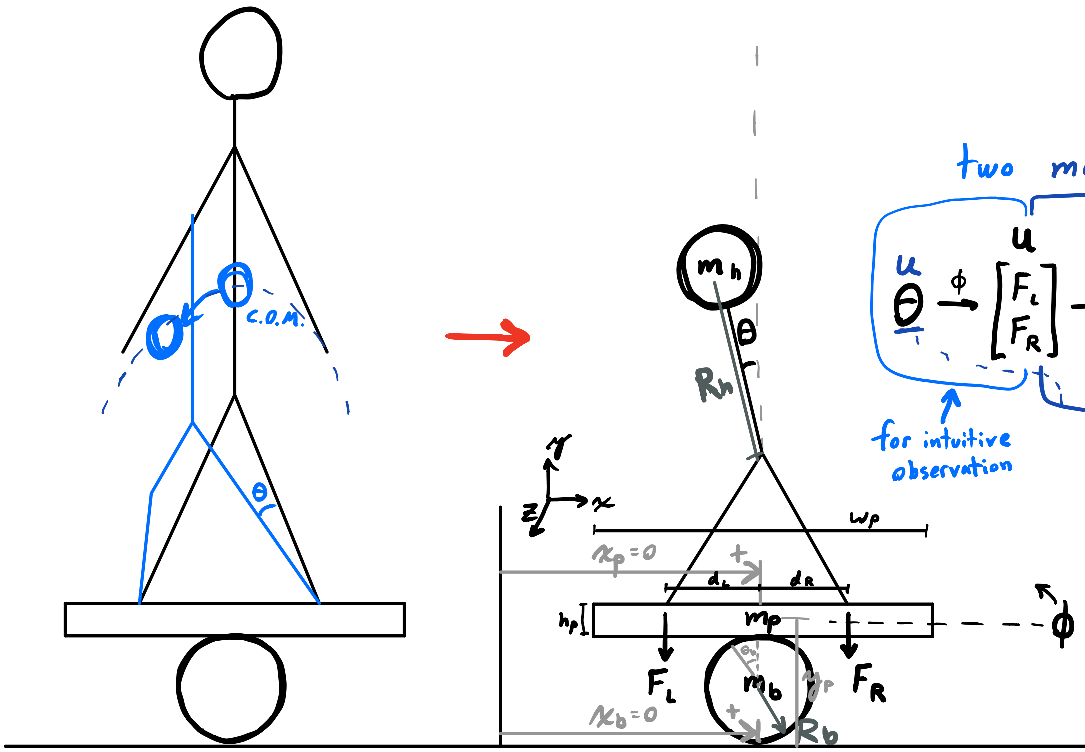
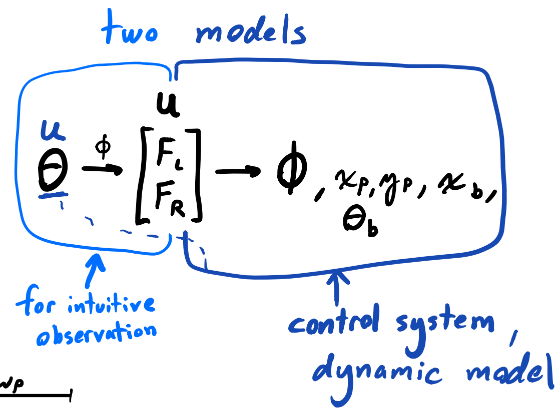




holonomic constraints:

$f_1(x_b,\theta_b) = x_b + R_b\theta_b = 0$ $\boxed{\ddot{x}_b = -R_b\ddot{\theta}_b}$

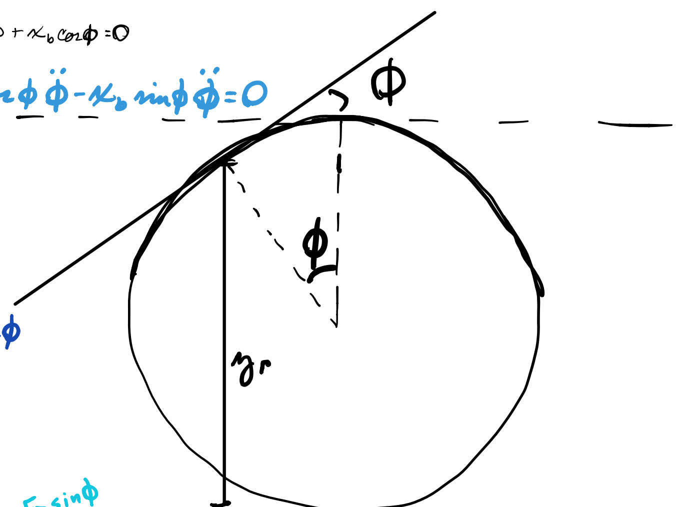
$f_2(y_p,\phi,\theta_b) = y_p - R_b(1+\cos\phi - \theta_b\sin\phi) = 0 \rightarrow y_p - R_b - R_b\cos\phi + R_b\theta_b\sin\phi = 0$

$2\cos\phi\,\dot{x}_b\dot{\phi} + R_b\cos\phi\,\dot{\phi}^2 - x_b\sin\phi\,\dot{\phi}^2 + \sin\phi\,\ddot{x}_b + \ddot{y}_p + x_b\cos\phi\,\ddot{\phi} + R_b\sin\phi\,\ddot{\phi} = 0$

$f_3(x_p,\phi,x_b,\theta_b) = x_p - x_b + R_b(\sin\phi + \theta_b\cos\phi) = 0 \rightarrow x_p - x_b + R_b\sin\phi + x_b\cos\phi = 0$

$-2\sin\phi\,\dot{x}_b\dot{\phi} - x_b\cos\phi\,\dot{\phi}^2 - R_b\sin\phi\,\dot{\phi}^2 - \ddot{x}_b + \cos\phi\,\ddot{x}_b + \ddot{x}_p + R_b\cos\phi\,\ddot{\phi} - x_b\sin\phi\,\ddot{\phi} = 0$

Each constraint reduces by one the number of degrees of freedom. If there are n dependent coordinates and m constraints we can use the constraints to reduce the number of degrees of freedom to $n-m$ and then apply $n-m$ Euler–Lagrange equations to solve the problem. This is conceptually simple, but may be fairly complicated in practice. A better approach is to keep all n variables and use the method of Lagrange multipliers. We now consider this method (often referred to as Lagrange's λ-method). We begin by taking the variation of the constraint equations, thus:



$\frac{\partial f_1}{\partial x_b} = 1$, $\frac{\partial f_1}{\partial \theta_b} = R_b$, $\frac{\partial f_2}{\partial y_p} = 1$, $\frac{\partial f_2}{\partial \phi} = R_b\sin\phi + R_b\theta_b\cos\phi$

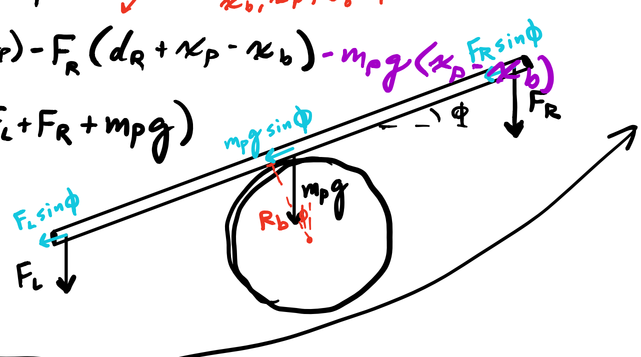
$\frac{\partial f_2}{\partial \theta_b} = R_b\sin\phi$, $\frac{\partial f_3}{\partial x_p} = 1$, $\frac{\partial f_3}{\partial \phi} = R_b\cos\phi - R_b\theta_b\sin\phi$, $\frac{\partial f_3}{\partial x_b} = -1$, $\frac{\partial f_3}{\partial \theta_b} = R_b\cos\phi$

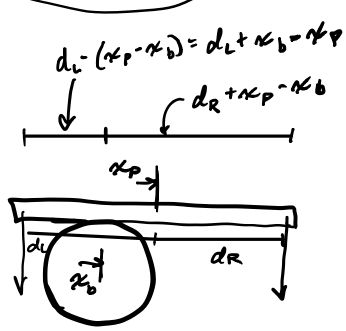
Handling external forces:

(on φ): $T_\phi = F_L d_{L\phi} - F_R d_{R\phi}$ need h.c.'s between x_b, x_p, θ_b, ϕ

$= F_L(d_L + x_b - x_p) - F_R(d_R + x_p - x_b) - m_p g(x_p - x_b)$

(on θ_b): $T_{\theta_b} = R_b\sin\phi(F_L + F_R + m_p g)$

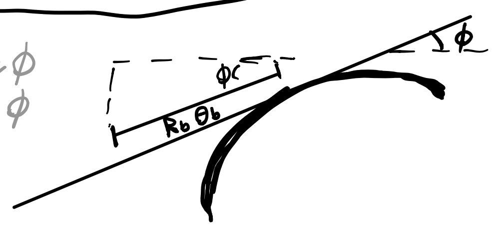




& y_p
x_p changes from φ effect
+ ball rolling(x_b)

$x_p = x_b - R_b\sin\phi - R_b\theta_b\cos\phi$

$y_p = R_b + R_b\cos\phi - R_b\theta_b\sin\phi$



$$I_{bz} = \frac{1}{2} m_b R_b^2$$

$$I_{pz} = \frac{1}{12} m_p \left(w_p^2 + h_p^2\right)$$

$$T = \frac{1}{2} m_p \left(\dot{x}_p^2 + \dot{y}_p^2\right) + \frac{1}{2} I_{pz} \dot{\phi}^2 + \frac{1}{2} m_b \dot{x}_b^2 + \frac{1}{2} I_{bz} \dot{\theta}_b^2$$

$$= \frac{1}{2} m_p \left[\dot{x}_p^2 + \dot{y}_p^2 + \frac{1}{12}\left(w_p^2 + h_p^2\right)\dot{\phi}^2\right] + \frac{1}{2} m_b \left[\dot{x}_b^2 + \frac{1}{2} R_b^2 \dot{\theta}_b^2\right]$$

$$V = m_p g y_p$$

$$L = T - V = \frac{1}{2} m_p \left[\dot{x}_p^2 + \dot{y}_p^2 + \frac{1}{12}\left(w_p^2 + h_p^2\right)\dot{\phi}^2\right] + \frac{1}{2} m_b \left[\dot{x}_b^2 + \frac{1}{2} R_b^2 \dot{\theta}_b^2\right] - \underline{m_p g y_p} \; ?$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = \lambda \frac{\partial f}{\partial s}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta}$$

$$q = \begin{bmatrix} x_p \\ y_p \\ \phi \\ x_b \\ \theta_b \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{x}_p} = m_p \dot{x}_p \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_p} = m_p \ddot{x}_p$$

$$\frac{\partial L}{\partial x_p} = 0 \longrightarrow m_p \ddot{x}_p = \lambda_3$$

$$\frac{\partial L}{\partial \dot{y}_p} = m_p \dot{y}_p \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_p} = m_p \ddot{y}_p$$

$$\frac{\partial L}{\partial y_p} = -m_p g \longrightarrow m_p \ddot{y}_p + m_p g = \lambda_2$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{12} m_p \left(w_p^2 + h_p^2\right)\dot{\phi} \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{12} m_p \left(w_p^2 + h_p^2\right)\ddot{\phi}$$

$$\frac{\partial L}{\partial \phi} = 0 \longrightarrow \frac{1}{12} m_p \left(w_p^2 + h_p^2\right)\ddot{\phi} = F_L\left(d_L + x_b - x_p\right) - F_R\left(d_R + x_p - x_b\right) + \lambda_2 R_b\left(\sin\phi + \theta_b \cos\phi\right)$$
$$+ \lambda_3 R_b\left(\cos\phi - \theta_b \sin\phi\right)$$

$$\frac{\partial L}{\partial \dot{x}_b} = m_b \dot{x}_b \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_b} = m_b \ddot{x}_b$$

$$\frac{\partial L}{\partial x_b} = 0 \longrightarrow m_b \ddot{x}_b = \lambda_1 - \lambda_3 \longleftarrow \lambda_1 = m_b \ddot{x}_b + m_p \ddot{x}_p$$

$$\frac{\partial L}{\partial \dot{\theta}_b} = \frac{1}{2} m_b R_b^2 \dot{\theta}_b \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_b} = \frac{1}{2} m_b R_b^2 \ddot{\theta}_b$$

$$\frac{\partial L}{\partial \theta_b} = 0 \longrightarrow \frac{1}{2} m_b R_b^2 \ddot{\theta}_b = \lambda_1 R_b + \lambda_2 R_b \sin\phi + \lambda_3 R_b \cos\phi$$

$$\frac{1}{2} m_b R_b^2 \underline{\ddot{\theta}_b} = -R_b^2 m_b \ddot{\theta}_b + R_b m_p \underline{\ddot{x}_p} + R_b \sin\phi \, m_p \underline{\ddot{y}_p} + R_b \sin\phi \, m_p g + R_b \cos\phi \, m_p \underline{\ddot{x}_p}$$

$\ddot{x}_b = -R_b\ddot{\theta}_b$

$2\cos\phi\,\dot{x}_b\dot{\phi} + R_b\cos\phi\,\dot{\phi}^2 - x_b\sin\phi\,\dot{\phi}^2 + \sin\phi\,\ddot{x}_b + \ddot{y}_p + x_b\cos\phi\,\ddot{\phi} + R_b\sin\phi\,\ddot{\phi} = 0$

$-2\sin\phi\,\dot{x}_b\dot{\phi} - x_b\cos\phi\,\dot{\phi}^2 - R_b\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{x}_b - \ddot{x}_b + \ddot{x}_p + R_b\cos\phi\,\ddot{\phi} - x_b\sin\phi\,\ddot{\phi} = 0$

$m_p\ddot{x}_p = \lambda_3$

$m_p\ddot{y}_p + m_p g = \lambda_2$

$\frac{1}{12}m_p(w_p^2 + h_p^2)\ddot{\phi} = F_L(d_L + x_b - x_p) - F_R(d_R + x_p - x_b) + \lambda_2 R_b(\sin\phi + \theta_b\cos\phi) + \lambda_3 R_b(\cos\phi - \theta_b\sin\phi)$

$m_b\ddot{x}_b = \lambda_1 - \lambda_3$

$\frac{1}{2}m_b R_b^2\ddot{\theta}_b = \lambda_1 R_b + \lambda_2 R_b\sin\phi + \lambda_3 R_b\cos\phi$

... see Kane's Method

$m_p\ddot{x}_p = \lambda_3$

$m_p\ddot{y}_p + m_p g = \lambda_2$

$$\frac{1}{12}m_p(w_p^2 + h_p^2)\ddot{\phi} = F_L(d_L + x_b - x_p) - F_R(d_R + x_p - x_b) + \lambda_2 R_b(\sin\phi + \theta_b\cos\phi) + \lambda_3 R_b(\cos\phi - \theta_b\sin\phi) \checkmark$$

$m_b\ddot{x}_b = \lambda_1 - \lambda_3$

$\frac{1}{2}m_b R_b^2\ddot{\theta}_b = \lambda_1 R_b + \lambda_2 R_b\sin\phi + \lambda_3 R_b\cos\phi \checkmark$

$2\cos\phi\,\dot{x}_b\dot{\phi} + R_b\cos\phi\,\dot{\phi}^2 - x_b\sin\phi\,\dot{\phi}^2 + \sin\phi\,\ddot{x}_b + \ddot{y}_p + x_b\cos\phi\,\ddot{\phi} + R_b\sin\phi\,\ddot{\phi} = 0$

$-2\sin\phi\,\dot{x}_b\dot{\phi} - x_b\cos\phi\,\dot{\phi}^2 - R_b\sin\phi\,\dot{\phi}^2 - \ddot{x}_b + \cos\phi\,\ddot{x}_b + \ddot{x}_p + R_b\cos\phi\,\ddot{\phi} - x_b\sin\phi\,\ddot{\phi} = 0$

$\ddot{x}_b = -R_b\ddot{\theta}_b$

$T_\phi = F_L(d_L + x_b - x_p) - F_R(d_R + x_p - x_b) \checkmark$

$T_{\theta_b} = R_b\sin\phi(F_L + F_R + m_p g) \checkmark$

Can I just make the state $\begin{bmatrix}\phi \\ \dot{\phi} \\ \theta_b \\ \dot{\theta}_b\end{bmatrix}$ and make all other variables reactionary?

"Cosmetic outputs" and redundancies

$x_p - x_b + R_b(\sin\phi + \theta_b\cos\phi) = 0 \rightarrow \underline{x_p = -R_b\theta_b - R_b(\sin\phi + \theta_b\cos\phi)}$

$x_b + R_b\theta_b = 0 \rightarrow x_b = -R_b\theta_b$

$y_p - R_b(1+\cos\phi - \theta_b\sin\phi) = 0 \leftarrow$ purely cosmetic (For animation)

$\rightarrow y_p = R_b(1+\cos\phi - \theta_b\sin\phi)$

System

$$\underbrace{\tfrac{1}{2} m_b R_b^2}_{J_b}\ddot{\theta}_b = R_b(\overbrace{F_L + F_R}^{F_T\ (\text{const})} + m_p g)\sin\phi$$

$$\underbrace{\tfrac{1}{12} m_p(w_p^2 + h_p^2)}_{J_p}\ddot{\phi} = F_L(d_L + x_b - x_p) - F_R(d_R + x_p - x_b) \cancel{- m_p}$$

$$= F_L(d_L + R_b(\sin\phi + \theta_b\cos\phi)) - F_R(d_R - R_b(\sin\phi + \theta_b\cos\phi))$$

Linearize

FYI, $F_L + F_R = F_T = \text{const.} \rightarrow F_L = F_T - F_R$

$$x = \begin{bmatrix}\phi \\ \dot\phi \\ \theta_b \\ \dot\theta_b\end{bmatrix} \rightarrow \dot{x} = \begin{bmatrix}\dot\phi \\ \ddot\phi \\ \dot\theta_b \\ \ddot\theta_b\end{bmatrix} = \underbrace{\begin{bmatrix}\dot\phi \\ \frac{\overset{F_T - F_R}{\cancel{F_L}}}{J_p}(d_L + R_b(\sin\phi + \theta_b\cos\phi)) - \frac{F_R}{J_p}(d_R - R_b(\sin\phi + \theta_b\cos\phi)) \\ \dot\theta_b \\ \frac{R_b}{J_b}(\overbrace{\cancel{F_L + F_R}}^{F_T} + m_p g)\sin\phi\end{bmatrix}}_{f(x,u)} = \begin{bmatrix}0 \\ 0 \\ 0 \\ 0\end{bmatrix}$$

$\underline{\phi_e = \dot\phi_e = 0}$

$\underline{\theta_b \text{ such that } \dot\theta_b = 0}$

$\rightarrow F_L(d_L + R_b\theta_b) = F_R(d_R - R_b\theta_b)$

$\rightarrow \cancel{F_T(d_L + R_b\theta_b) = F_R(d_R + d_L)}$

$\rightarrow F_{Re} = \dfrac{F_T(d_L + R_b\theta_b)}{d_R + d_L}$

State-Space Jacobian Linearization

1. $f(x_e, u_e) = 0$

make a simple domain shift...

drop it like it's...

2.
$$f(x,u) \approx \cancel{f(x_e,u_e)}^{\,0} + \underbrace{\frac{\partial f}{\partial x}\bigg|_e}_{A}\underbrace{(x - x_e)}_{\tilde{x}\to x} + \underbrace{\frac{\partial f}{\partial u}\bigg|_e}_{B}\underbrace{(u - u_e)}_{\tilde{u}\to u} + \cancel{H.O.T.}$$
$$h(x,u) \approx h(x_e,u_e) + \underbrace{\frac{\partial h}{\partial x}\bigg|_e}_{C}(x - x_e) + \underbrace{\frac{\partial h}{\partial u}\bigg|_e}_{D}(u - u_e) + H.O.T.,$$

where the Jacobian matrices are defined as

variables

functions

$$\frac{\partial f}{\partial x} \triangleq \begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{pmatrix}$$
$$\frac{\partial f}{\partial u} \triangleq \begin{pmatrix}\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m}\end{pmatrix}$$
$$\frac{\partial h}{\partial x} \triangleq \begin{pmatrix}\frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n}\end{pmatrix}$$
$$\frac{\partial h}{\partial u} \triangleq \begin{pmatrix}\frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m}\end{pmatrix}.$$

n = num_states
m = num_inputs
p = num_outputs

A: $n \times n$
B: $n \times m$
C: $p \times n$
D: $p \times m$

$$\dot{x} = f(x,u) \approx \cancel{f(x_e,u_e)}^{\,0} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{F_T R_b}{J_p} & 0 & \frac{F_T R_b}{J_p} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{(F_T + g m_p)R_b}{J_b} & 0 & 0 & 0\end{bmatrix}}_{A}\tilde{x} + \underbrace{\begin{bmatrix} 0 \\ -\frac{d_L + d_R}{J_p} \\ 0 \\ 0\end{bmatrix}}_{B}\tilde{F}_R$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0\end{bmatrix}}_{C}\tilde{x}$$